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A. C. Palmer, et al

Brown University

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* Engineering Department, Cambridge University, England

** Division of Engineering, Brown University, Providence, R.I., USA

ABSTRACT

In heavily over-consolidated clays there is a marked peak in the observed relation between shear stress and shear strain. As the strain increases, the stress falls from a peak to a much smaller residual stress. Slopes made from such a clay often fail progressively many years after construction. Sliding occurs on a concentrated slip surface, and it is found that the mean resolved shear stress on that surface is markedly less than the peak shear strength. Concepts from fracture mechanics, and in particular the J-integral, are used to derive conditions for the propagation of a concentrated shear band of this kind. The results indicate the presence of a strong size effect, which has important implications for the use of models in soil mechanics. An elastic analysis makes it possible to determine the size of the end zone in which the shear stress on the shear band falls to its residual value. An attempt is made to assess the possible sources of the time-dependence governing propagation speed of the shear band. They include pore-water diffusion to the dilating tip of the band (which governs the rate at which local strength reductions can occur), visco-elastic deformation of the clay (which allows a gradual build-up of strain concentration at the tip of the band), and the weathering break-down of diagenetic bonds.

INTRODUCTION

A striking feature of landslides and foundation failures in over-consolidated clay soils is that most of the deformation is concentrated in narrow zones which lie between regions which appear hardly to deform at all. The concept of a concentrated "slip surface" or "failure surface" appeared early in the history of soil mechanics, and much of soil mechanics theory is based on it. Characteristically, someone analysing a slope postulates a mode of failure in which one or more concentrated slip surfaces form, supposes a limiting shear strength to act across these surfaces, and considers the equilibrium of the blocks into which the surfaces divide the slope. The theory of plasticity gives some support to this approach, and, indeed, it would be just as valid from that point of view if slip surfaces did not actually occur.

Much less attention has been given to problems of the initiation and development of slip surfaces. In this paper we examine the consequences of a simple model for the growth of these surfaces, which we call "shear bands". Among other things, we hope to throw light on some apparent paradoxes of the conventional approach to slope failure, in particular the observation that "progressive failure" (Bjerrum, 1967a) can occur even though the mean shear stress on the observed failure surface is substantially less than the shear stress the clay can withstand.

We take as our starting point an observation of what happens when overconsolidated clay is tested in a shear box, as illustrated schematically in Figure 1a. It is the simplest apparatus that has been used to study shear in soils, and the oldest, having been used by Coulomb. The vertical load is kept constant. The observed relation between the relative horizontal displacement between the upper and lower halves of the box and the applied

shear force is as shown in Figure 1b (see, for example, Skempton, 1964). A peak force is reached at quite a small displacement. After the peak has been passed the deformation is concentrated in a relatively narrow shear band, less than 1 mm thick. The force required to produce further relative movement then falls continuously, and asymptotically approaches a value corresponding to a "residual" mean shear stress.

This observation prompts us to consider a model of soil deformation in which relative shear displacements can occur in concentrated shear bands, the relation between relative displacement δ and shear stress τ across the band being like that shown in Figure 1c. Outside the shear band the soil deforms continuously, and obeys conventional stress-strain relations. The peak shear strength is τ_p, and the residual shear strength τ_r. They will both depend on the prevailing effective normal stress across the band.

The assumed model of a shear band in soil has much in common with cohesive-force models of tensile cracks (Barenblatt, 1962; Dugdale, 1960; Bilby, Cottrell and Swinden, 1963). In particular, we shall follow the development by Rice (1968 a,b) of a unified approach to such models based on the J-integral. Skempton (1964) and Bishop (1968) have suggested that fracture mechanics concepts might throw light on progressive failure, and Bjerrum (1967a) has discussed a model of progressive failure in terms of stress concentrations at the tip of a slip surface. The microstructure of shear bands has been investigated by Morgenstern and Tchalenko (1967 a,b). In this paper we leave aside the question of the detailed structure of real shear bands, and that of the localisation of deformation into shear bands. Instead we consider the shear band simply as a surface of discontinuity on which there exists a definite relationship between shear stress and relative displacement.

A SIZE EFFECT

An immediate consequence of our model is that size effects will occur. The assumption of a relation between shear stress and shear displacement introduces a characteristic length into the material description. This length will necessarily enter a prediction of final failure conditions in relation to some characteristic dimension describing the geometry of the soil system. Consider, for example, a natural slope and a small geometrically similar model of the slope, the model and the natural slope being made of the same material. Suppose that there are no shear bands in either. Then, if the model is loaded by an appropriately scaled gravity field, as in a centrifuge, the conditions for full similarity of stress and strain fields can be met. If, on the other hand, the model and the natural slope have geometrically similar shear bands, then the similarity conditions are no longer satisfied. If the strain fields were indeed similar, then, recalling that displacements are integrals of strain with respect to distance, we must conclude that at a point on a band the natural slope would have a relative displacement δ greater than the relative displacement at the corresponding point on the small model. The displacements at similar points would be in the ratio of the scale of the model and the natural slope. However, τ is a fixed decreasing function of δ, and this means that τ at any point along the band in the natural slope would be less than τ at the similar point in the model.

This last result is of course inconsistent with the similarity of strains outside the band. However, the conclusion is clear that the large slope will have larger δ values and hence smaller τ values than does the small model at similar points along the band. Thus, for example, it is possible that in a natural slope the shear stress could be

near its residual value everywhere except for a localized zone near the tip of a band, whereas for a sufficiently small geometrically similar model the shear stress would have barely decreased from the peak value along the entire (but small) length of the band.

Bishop (1971) has proposed that the Skempton residual factor, measuring the amount of fall from peak toward residual strength, should be considered a function of position along the band. This is consistent with our present model in that the relative sliding δ will generally be an increasing function, and hence τ a decreasing function, of distance from the tip of the band. Bishop pointed out that a size effect would result from the requirement of a certain displacement on a slip surface before the residual stress is reached.

We shall attempt quantitative estimates of these size effects in the following sections, but only when the simplicity of the shear band geometry lends confidence to the accompanying analysis. Specifically, the examples to follow will all deal with straight shear bands propagating in their own plane. Further, the typically jointed structure of overconsolidated clays could of itself lead to a size effect as, for example, Marsland (1972) has proposed. We do not have a way of including this effect in the model, except to say that the stress-strain relations employed outside the shear band should be those appropriate to the actual jointed material. Thus the size effects under consideration here are solely those due to the progressive degradation, with increasing δ, of the shear strength of material within the slip surface.

THE J-INTEGRAL

In the following sections we derive conditions for the propagation of

a shear band. Our most important analytic tool is the J-integral of crack mechanics (Rice, 1968 a,b). Define Cartesian axes x_1 and x_2 (Figure 2) so that a straight shear band lies parallel to the x_1-axis, and suppose plane-strain deformation to occur in the x_1, x_2 plane. Let the stress-strain relation of the material outside the band be such that the stress work integral

$$W(\varepsilon_{pq}) = \int_0^{\varepsilon_{pq}} \sigma_{ij}\, d\varepsilon_{ij} \tag{1}$$

at any strain ε_{pq} experienced by the soil is independent of the strain path. An elastic material clearly obeys this condition. The material properties, the body forces, and any prestress existing in the reference state, can depend on x_2 but not on x_1. Let Γ be a curve in the x_1, x_2 plane which starts at a point P^- on the lower surface of the shear band, goes round the tip of the band, and ends at a point P^+ on the upper surface, where P^+ and P^- coincide in the unstrained reference state. Let the outward-pointing unit normal vector to Γ have components n_i, let u_i be the components of displacement, and let T_i be the surface tractions across Γ, related to the stress components σ_{ij} by

$$T_i = \sigma_{ij}\, n_j \tag{2}$$

Further, let f_i be the components of body force per unit volume. The J-integral is then defined by

$$J_P = \int_\Gamma \left[(W - f_i u_i)dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds \right] \tag{3}$$

where ds is an element of arc length of Γ.

This integral is useful because its value is independent of the path

of integration Γ, and depends only on the end-points P^+ and P^-. The dependence on the end points follows only because stress is transmitted across the band, and would not occur for a freely slipping band or for an open tensile crack. A proof of path independence has been given by Rice (1968 a) in the case where there are no body forces f_i; the extension to the proof to include body forces is trivial.

Sometimes we shall want to apply the integral to inelastic materials for which the stress work integral $W(\epsilon_{pq})$ is not independent of strain path. It turns out that J is still independent of the path Γ, as long as the difference between the values of W at two points (x_1', x_2') and (x_1'', x_2') on a line parallel to the x_1-axis is defined by

$$\int_{x_1'}^{x_1''} \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial x_1} dx_1 ,$$

the integral along the line between the two points. It is only this difference that contributes to the J-integral.

We now let the path Γ have a particular form. Suppose it to follow the lower surface of the band from P^- to the tip of the band, and to return to P^+ along the upper surface. Then dx_2 is zero along the whole path, and so the first terms of the integral vanish. Across the band u_2 is continuous, and therefore $\partial u_2/\partial x_1$ is continuous, whereas T_2 at a point on the upper surface is equal and opposite to T_2 at the corresponding point on the lower surface, and so the $T_2 \partial u_2/\partial x_1$ term makes no contribution to the integral. Hence, using (2), we have

$$J_P = \int_\Gamma \sigma_{21} \frac{\partial u_1}{\partial x_1} dx_1 \qquad (4)$$

for this choice of Γ. Across the band σ_{21} must be continuous; if u_1^+

and u_1^- are displacements on the upper and lower surfaces, and δ is the relative displacement $u_1^+ - u_1^-$, then, since the upper and lower surfaces are traversed in opposite directions, equation (4) gives

$$J_P = \int_T^P \sigma_{21} \frac{\partial}{\partial x_1} (u_1^+ - u_1^-) \, dx_1 = \int_T^P \tau \frac{\partial \delta}{\partial x_1} \, dx_1 \tag{5}$$

the integrals being taken from the band tip T to P, and τ being the shear stress across the band.

Our model of shear bands asserts that there is a fixed relationship between, τ and δ, at least so long as the band does not unload and become inactive; τ is then a single-valued function $\tau(\delta)$ of δ, and we can write (5) as

$$J_P = \int_0^{\delta_P} \tau(\delta) \, d\delta \tag{6}$$

Outside an end region close to the band tip, the relative displacement is large enough to reduce τ_P to the residual stress τ_r (Figure 1c). It is then convenient to divide the integral in (6) into a part corresponding to a residual stress and a remainder contributed by the difference between the shear stress and the residual stress at small displacements, so that if P lies outside the end zone

$$J_P - \tau_r \delta_P = \int (\tau - \tau_r) \, d\delta \tag{7}$$

and $J_P - \tau_r \delta_P$ is independent of P. The integral in (7) denotes the cross-hatched area in Figure 1c. A characteristic displacement $\bar{\delta}$ can be defined by

$$\int (\tau - \tau_r)\, d\delta = (\tau_p - \tau_r)\, \bar{\delta} \qquad (8)$$

Shear tests on overconsolidated clay reported by Skempton (1964) and Skempton and Petley (1968) are consistent with values of $\bar{\delta}$ between 2 and 10 mm.

What we shall next do is to exploit the path-independence of the integral. Equation (7) gives the value of $J_p - \tau_r \delta_p$ for an active shear band. If we evaluate $J_p - \tau_r \delta_p$ along a different wider path with the same end points, we find it to be an increasing function of the applied loads. When the loads become large enough for $J_p - \tau_r \delta_p$ to reach its critical value, the band becomes active, and will propagate if the loads are increased any further. Equation (7) can be thought of as an energy balance of the Griffith type, if the end zone remains small or propagates unchanged (in the sense that an observer moving with the band tip always sees the same distribution of strain). The $J_p - \tau_r \delta_p$ can be interpreted as the energy surplus made available per unit area of advance of the band, this surplus being the excess of the work input of the applied forces over the sum of the net energy absorbed in deforming material outside the band and the frictional dissipation against the residual part τ_r of the slip resistance within the band. Accordingly, equation (7) asserts that for propagation to occur this net energy surplus must just balance the additional dissipation in the end region against shear strengths in excess of the residual. This interpretation is developed in the Appendix.

A SLIP SURFACE IN A LONG SHEAR APPARATUS

Consider a long shear apparatus of the kind shown in fig.3. This contains a layer of overconsolidated soil of height h between two rigid boundaries. The lower boundary is fixed while the upper boundary is displaced horizontally by an amount u_b. A shear band is initiated from

the left boundary, possibly with the aid of a local stress concentration from a cut or notch, and has now extended into the interior of the specimen. We shall use the J-integral to find the criterion for continuing propagation, on the assumption that the apparatus is very long compared to its height and that the end region of the shear band is far from either the right or left boundary of the specimen. Under these conditions h is the only significant dimension, and the apparatus may be considered of infinite horizontal extent. The weight of the soil may be neglected. Note that in the region far ahead of the tip of the band (but not too close to the right hand boundary, where end effects may appear) the soil is in a state of homogeneous shear strain and stress

$$2\varepsilon_{12} = \gamma_o \equiv \frac{u_b}{h} \quad \text{and} \quad \sigma_{12} = \tau_o \tag{9}$$

where τ_o is the shear stress corresponding to the shear strain γ_o, and we assume of course that $\tau_0 < \tau_p$. Likewise, far to the left of the tip, but not too close to the left-hand boundary, there will also be a homogeneous state in the soil above and below the slip surface:

$$2\varepsilon_{12} = \gamma_r \quad \text{and} \quad \sigma_{12} = \tau_r \tag{10}$$

where τ_r is the residual shear strength which is acting along the band in that region.

Consider the choice of point P and path Γ illustrated in fig.3. The relative displacement at P is

$$\delta_P = \gamma_o h - \gamma_r h/2 - \gamma_r h/2 = h(\gamma_o - \gamma_r) \tag{11}$$

where the first term is the imposed boundary displacement and the two subtracted terms $\gamma_r h/2$ represent that portion of the imposed boundary displacement taken up by soil deformation in the regions above and below the band. The integrand of J_p (eq.3) will vanish all along the rigid boundaries, because dx_2 and $\partial u_i/\partial x_1$ vanish there. Likewise, $\partial u_i/\partial x_1$

vanishes in the homogeneously strained regions far to the left and right of the tip, so that for the path Γ

$$J_P = \int_\Gamma W \, dx_2 = h\, W(\gamma_o) - h\, W(\gamma_r) \tag{12}$$

Here we use the notation $W(\gamma)$ for the energy density in a region under homogeneous shear strain γ. Thus we have obtained the "driving force" term in the propagation criterion (eq.7) as

$$J_P - \tau_r\, \delta_P = h\left[W(\gamma_o) - W(\gamma_r) - \tau_r(\gamma_o - \gamma_r)\right] \tag{13}$$

This result reinforces the energetic interpretation of $J_P - \tau_r\, \delta_P$ given earlier. Consider the energy changes which result when the slip surface advances a distance $\Delta\ell$ while the boundary remains fixed. There is no work input from boundary forces. The loss in deformation energy can be computed by noting that this slip surface advance essentially allows an area of material $h\Delta\ell$ to reduce its energy density from $W(\gamma_o)$ to $W(\gamma_r)$, and is

$$h\Delta\ell \left[W(\gamma_o) - W(\gamma_r)\right]$$

The work dissipated in the band against the residual part of the shear strength is the same as that dissipated in sliding a segment $\Delta\ell$ of the band a distance equal to the uniform slip displacement $h(\gamma_o - \gamma_r)$ far from the tip, namely

$$\tau_r\, h(\gamma_o - \gamma_r)\, \Delta\ell$$

Thus the net energy surplus, available for work against that part of the strength in excess of the residual value, is just the sum of these two terms, which we see to be $(J_P - \tau_r\, \delta_P)\, \Delta\ell$ as expected.

To interpret the driving force in terms of the shear stress-strain curve $\tau = \tau(\gamma)$, fig.4a, note that

$$W(\gamma_o) - W(\gamma_r) = \int_{\gamma_r}^{\gamma_o} \tau(\gamma)\, d\gamma \tag{14}$$

so that (13) becomes

$$J_p - \tau_r\, \delta_p = h \int_{\gamma_r}^{\gamma_o} \left[\tau(\gamma) - \tau_r\right] d\gamma \tag{15}$$

The graphical interpretation of this driving force is as h times the shaded area in fig.4a. If the material is linear elastic, or approximated by a linear relation of the form

$$\tau_o - \tau = G(\gamma_o - \gamma) \tag{16}$$

in the strain range of interest, where G is a shear modulus, the driving force may be written as

$$J_p - \tau_r\, \delta_p = \frac{h}{2G} (\tau_o - \tau_r)^2 \tag{17}$$

In general, however, the soil will not be perfectly elastic and will unload along a different curve from that for loading, as in fig.4b. It is difficult to treat this in the same precise manner. But, by recalling the definition of W as an integral in the x_1 direction for inelastic materials, we see that an approximately correct answer can be obtained if we define $W(\gamma_o) - W(\gamma_r)$ of eq.14 from the unloading stress-strain curve as in fig.4b. This is because the integral in the x_1 direction essentially traces deformation states encountered as the material outside the band transforms from the homogeneous stress state τ_o, existing far to the right of the tip, to the residual state existing far to the left. Hence it seems appropriate to adopt eqs.15, 17 for the driving force in this case, provided that the unloading stress-strain curve is used to identify it as h times the

shaded area in fig.4b, and that in the linear approximation the shear modulus G is that governing unloading. This same choice also seems appropriate from an energetic viewpoint, in that it is the energy made available upon unloading which can contribute to further advance of the band. Time effects due to creep or diffusion may also play a role in determining the stress-strain curve to be chosen, and we discuss this subsequently after an estimate of the end zone size is available.

In any event, for some suitably chosen G in the linear approximation, the propagation criterion becomes

$$\frac{h}{2G}(\tau_o - \tau_r)^2 = \int(\tau - \tau_r)\, d\delta \tag{18}$$

or, if the additional end region energy absorption is written as in (eq.8),

$$\frac{\tau_o - \tau_r}{\tau_p - \tau_r} = \sqrt{\frac{2G}{\tau_p - \tau_r}\frac{\bar{\delta}}{h}} \tag{19}$$

This reveals the size-effect on the propagation stress level τ_o: the greater the height h of the layer the smaller the stress excess $\tau_o - \tau_r$ required for propagation. In fact, there is also an abrupt cut-off because the left side of this equation cannot exceed unity. Thus if

$$h < \frac{2G}{\tau_p - \tau_r}\bar{\delta} \quad , \tag{20}$$

the propagation condition cannot be met before the stress τ_o induced in the layer reaches the peak strength and more-or-less simultaneous failure of the layer occurs. That is, for a sufficiently thin layer, the energy which may be stored by a stress as large as the peak value will still be insufficient to supply the required energy surplus in a unit advance of the shear band.

Wroth (1972) has noted that for overconsolidated London clays $G/\tau_p \approx 50$.

Thus the critical layer height, below which failure occurs at $\tau_o = \tau_p$, is

$$h_{cr} = \frac{2G}{\tau_p - \tau_r}\,\bar{\delta} = 100\,\frac{\tau_p}{\tau_p - \tau_r}\,\bar{\delta} \qquad (21)$$

If we take 2 for the stress ratio and 5 mm for $\bar{\delta}$, as typical values, the critical height turns out to be 1 m. This is catastrophic from the point of view of laboratory experimentation, for the height is unreasonably large as a lower limit to the required specimen size for studying slip surface extension. Of course, 1 m is not a large dimension in typical field failures. (It should be noted that Wroth's ratio is based on the G for loading; the preferred G governing unloading must be higher and this will increase the numerical factor in eq.(21) in proportion).

SLIP SURFACE FROM A STEP IN A SLOPE

Referring to fig.5a, we now consider a long flat slope of inclination angle α into which a step of height h has been cut. A shear band of length ℓ emanates from the base of the cut in a direction parallelling the ground surface. We wish to obtain expressions for the driving force on the band and, in particular, for the propagation criterion. It is clear that this case presents in elementary form some of the factors likely to be important in failure of a natural slope. Nevertheless, a precise analysis is difficult and we here present an approximation for the case in which the band length is large compared to the layer thickness and to the size of the end region. Under such conditions most of the energy transfer during shear band extension will be due to gravitational work on downslope movements of the layer and to deformations of the layer from changes in the normal stress acting parallel to the slope surface.

The stress state σ^o_{ij} existing <u>before</u> the cut is made is supposed to

depend only on depth from the slope surface. The corresponding infinite slope equations for the adopted coordinate system, fig.5a, are

$$\sigma_{22}^{o} = -\rho g \, x_2 \cos\alpha, \quad \sigma_{21}^{o} = \rho g \, x_2 \sin\alpha, \quad \sigma_{11}^{o} = f(x_2) \tag{22}$$

where ρ is the average density for depth x_2 and where the last of these is intended to indicate that σ_{11}^{o} is undertermined by equilibrium considerations alone. We shall be interested in the average value of σ_{11} over depth h,

$$\bar{\sigma}_{11} = \frac{1}{h} \int_0^h \sigma_{11} \, dx_2 \tag{23}$$

and shall write $p^{o} = -\bar{\sigma}_{11}^{o}$ for the average lateral earth pressure existing prior to introduction of the cut; p^{o} may reflect a normal lateral pressure effect, or possibly some augmented pressure due to the weathering break-down of diagenetic bonds (Bjerrum, 1967a). We shall write the gravitationally induced shear stress on the prospective failure plane as

$$\tau_g = (\sigma_{21}^{o})_{x_2 = h} = \rho g \, h \sin\alpha \tag{24}$$

All displacements and strains will be measured from zero in the pre-stressed state existing before the cut is made.

To evaluate the driving force we choose the point P and path Γ shown in fig.5a. Further, from what has been said above, we will neglect any displacement or straining in the base material below the slip surface ($x_2 > h$) since the dominant deformations and energy transfers may be assumed to occur in the sliding layer. Hence the J integrand may be assumed to vanish along that portion of Γ through the base material. It also vanishes far up the slope where there has been no displacement from the prestressed state. We are left only with the portions of Γ along the inclined ground

surface and the surface of the cut. Since dx_2 and the surface traction vanish along the former, and the surface traction also vanishes along the latter, we are left with

$$J_p = -\int_0^h (W + \rho g \sin\alpha \, u_1 - \rho g \cos\alpha \, u_2)_{x_1=0} \, dx_2 \tag{25}$$

Since the reference state for strains is that of the state under pre-stresses σ^o_{ij}, W is here to be interpreted as the energy recovered during deformation from the prestressed state to the state of zero transverse stress existing at the cut surface.

When the layer is long in comparison to its height we may assume that its deformation is essentially a one-dimensional displacement in the negative x_1 direction, and that at any point the magnitude of this displacement is the same as the relative sliding δ at the same value of x_1 : $u_1 = -\delta(x_1)$ Thus

$$J_p = -\overline{W}\, h + (\rho g \, h \sin\alpha)\, \delta_p = -\overline{W}\, h + \tau_g \, \delta_p \tag{26}$$

where $\overline{W}$ is the thickness average energy density at the end of the slope. This is defined from the stress-strain curve relating the thickness average stress $\bar{\sigma}_{11}$ in the layer to the strain $\bar{\varepsilon}_{11}$:

$$W = \int_{\bar{\sigma}_{11} = -p_o}^{\bar{\sigma}_{11} = 0} \bar{\sigma}_{11}(\bar{\varepsilon}_{11}) \, d\bar{\varepsilon}_{11} \quad , \tag{27}$$

and is the negative of the shaded area identified in fig.5b. The driving force term is therefore

$$J_p - \tau_r \, \delta_p = (\tau_g - \tau_r)\, \delta_p - h\, \overline{W} \tag{28}$$

If we further recall the assumption that the end region is small, so that

$\sigma_{21} = \tau_r$ along nearly the entire length of the shear band, then it is clear from overall equilibrium in the x_1-direction that $\bar{\sigma}_{11}$ is given by

$$\bar{\sigma}_{11}\, h = (\tau_g - \tau_r)\, x_1$$

Thus

$$(\tau_g - \tau_r)\,\delta_p = (\tau_g - \tau_r)\int_{x_1=0}^{x_1=\ell} \bar{\varepsilon}_{11}\, dx_1 = h\int_{x_1=0}^{x_1=\ell} \bar{\varepsilon}_{11}\, d\bar{\sigma}_{11}$$

$$= h\int_{\bar{\sigma}_{11}=0}^{\bar{\sigma}_{11}=(\tau_g-\tau_r)\ell/h} \bar{\varepsilon}_{11}\, d\bar{\sigma}_{11} \qquad (29)$$

and the corresponding area is also shaded in fig.5b.

From eq.(28) it is clear that the driving force is just h times the sum of the two shaded areas, and the final result is therefore

$$J_p - \tau_r\,\delta_p = h\int_{-p^o}^{(\tau_g-\tau_r)\ell/h} \bar{\varepsilon}_{11}(\bar{\sigma}_{11})\, d\bar{\sigma}_{11} \qquad (30)$$

From the energetic point of view, the lower cross-hatched area represents the energy which is recovered in a unit advance of the shear band due to relief of the transverse pressure p^o, whereas the upper cross-hatched area represents the excess of work input by the gravity forces over the dissipation against the residual shear strength.

If the stress-strain curve for the layer is represented in the linear form

$$\bar{\sigma}_{11} = -\,p^o + E'\,\bar{\varepsilon}_{11} \qquad (31)$$

where E' is an overall elastic modulus for the layer under the assumed plane strain conditions, then the driving force expression and propagation criterion take the form

$$J_p - \tau_r \delta_p = \frac{h}{2E'}\left[(\tau_g - \tau_r)\,\ell/h + p^o\right]^2 = \int(\tau - \tau_r)\,d\delta \qquad (32)$$

Also, with the notation of eq.(8), this may be put in the dimensionless form

$$\frac{(\tau_g - \tau_r)\,\ell/h + p^o}{\tau_p - \tau_r} = \sqrt{\frac{2E'}{\tau_p - \tau_r}\frac{\bar{\delta}}{h}} \qquad (33)$$

It is perhaps of special interest to note that even if the slope angle is such that the gravitationally induced shear stress equals the residual strength (i.e., $\tau_g = \tau_r$), so that Skempton's residual factor is zero, it is still possible that the energy recovered by relief of the initial pressure p^o could be adequate to drive the shear band. This was suggested by Bjerrum (1967) and the corresponding special case of the above formula gives a quantitative estimate of the required initial pressure.

We shall consider this case a little further in the subsequent discussion of possible sources of time effects. It must be remembered, however, that there have been several approximations made in our treatment. They seem to be appropriate when the band is indeed long and when the end region occupies only a small fraction of the total length. However a more refined analysis, based perhaps on a finite element analysis of the soil outside the band, with the τ, δ relation as a boundary condition, will be necessary if the exact nature of the approximations is to be examined, and if the model is to be extended to other cases involving, say, non-planar slip surfaces.

LINEAR ELASTIC ANALYSIS WITH SMALL END REGION

Henceforth we consider the soil outside the shear band to be homogeneous,

isotropic, and linear elastic, and we consider only cases for which the end region length (in which the shear stress falls to its residual value) is small in comparison to all geometric dimensions such as overall band length, layer height, etc.. We shall, indeed, first examine the limiting case in which the end region is taken to be infinitely small, so that the shear band carries the residual strength along its entire length. In this idealization stresses predicted will become infinite at the tip of the band. We shall identify the dominant terms in this singular stress distribution near the tip, and then proceed to take the view that an end region of small but finite size may be considered to be embedded in a local stress field for which the dominant terms set the outer field boundary conditions. That is to say, the dominant stress terms as obtained from the simpler model with no end region incorporate the actual effect of applied loadings and overall geometry of the failing soil mass on the deformations in the end region. A similar approach is much used in fracture mechanics and indeed provides the rationale for use of elastically computed crack tip stress fields in semi-ductile metals failing under conditions of a small plastic region at the crack tip. The intensity of the singularity is then expressed by a stress-intensity factor, calculated from a complete elastic solution which in turn depends on the applied loads and the crack geometry. This solution is not valid in the plastic region at the crack tip. However, it is known that when the plastic region is small compared to other pertinent geometric dimensions, proper characterization is obtained if the elastic singularity is seen as setting outer field boundary conditions. The applied loads and geometrical dimensions influence the stress state in the crack tip plastic region only insofar as they enter the expression for the elastically computed stress intensity factor. This is the "small scale yielding" formulation of crack tip plasticity as discussed by Rice (1968a, 1968b).

We wish to obtain the form of the stress distribution near the tip of a shear band which is assumed to carry a constant or smoothly varying residual strength τ_r along its length. The form is already known for a straight slit under plane strain shear loadings relative to the crackline. As it happens, the loadings induce no opening separations of the crack surfaces to that the same model describes a freely slipping shear band. We have only to adjust these known results by adding on terms to represent the shear and normal stresses transmitted across the band. Upon adapting the crack formulae (e.g. Rice 1968b) in this way, we therefore find that the stress distribution at the tip of a shear band takes the characteristic form (referred to polar coordinates R,θ of fig.6)

$$\begin{aligned}
\sigma_{12} &= (2\pi R)^{-1/2} K \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \tau_r + \ldots \\
\sigma_{22} &= (2\pi R)^{-1/2} K \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \sigma_n + \ldots \qquad (34) \\
\sigma_{11} &= -(2\pi R)^{-1/2} K \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] + \sigma_t + \ldots
\end{aligned}$$

The dots represent other terms, all of which vanish at $R = 0$, in a complete expansion of the stress field in powers of R; σ_n is the normal stress transmitted across the band and σ_t is the transverse stress acting along the line directly ahead of the band. In addition to these constant stress terms, however, there is a singular part of the stress field which becomes infinite as $R^{-1/2}$ and which has a characteristic angular distribution. The strength of the singular term is given by the "stress intensity factor" K, which will be a function of the loadings and geometrical dimensions of the soil mass containing the shear band. For example, the K factor for a shear band of length ℓ in a body under the remote shear stress τ_∞ ($> \tau_r$) as in fig.7 is (e.g. Rice 1968b)

$$K = (\tau_\infty - \tau_r)\sqrt{\pi\ell/2} \qquad (35)$$

Likewise, for the shear band in the long shear apparatus of fig.3

$$K = (\tau_o - \tau_r) \sqrt{h/(1 - \nu)} \tag{36}$$

where ν is the Poisson ratio, and for the shear band emanating from the step in the slope, fig.5a,

$$K = \left[(\tau_g - \tau_r) \frac{\ell}{h} + p^o\right] \sqrt{h/2} \tag{37}$$

These assume that the residual stress is indeed activated all along the shear band.

The displacement field associated with the above stress state results in a slip displacement

$$\delta = u_1^+ - u_1^- = \frac{4\ (1+\nu)\ K}{G} \left(\frac{R}{2\pi}\right)^{1/2} + \ldots \tag{38}$$

The J-integral can be evaluated directly , making use of the corresponding displacement field, and is

$$J_P - \tau_r\ \delta_P = \frac{1-\nu}{2G} K^2 \tag{39}$$

which is the well-known Irwin formula for the energy release rate. It is in fact through this formula and the earlier direct evaluations of $J_P - \tau_r\ \tau_P$ that eqs.(36,37) are obtained. The result needs no detailed proof here, for it has already been remarked that $J_P - \tau_r\ \delta_P$ is independent of the location of point P. Further it is clear that if point P and the path Γ are taken very near the tip, only the $R^{-1/2}$ singular terms can contribute, and in that instance the calculation is the same as that of the J-integral in crack theory for which eq.(39) is the known result. Remembering however that the dominant stress terms of this analysis give the outer field boundary conditions for the case of a small end zone, we may again assert that the propagation criterion is given in terms of the τ,δ curve by eq.(7), with the driving force expressed as in eq.(39). Hence the propagation criterion is

$$\frac{1-\nu}{2G} K^2 = \int(\tau - \tau_r)\, d\delta \tag{40}$$

where K is determined from an analysis which neglects the end zone in the manner discussed above. This is a quite general procedure for dealing with small end zones, although the proper modifications to account for inelasticity outside the band cannot be stated with any generality.

In particular, on the assumption of a small end zone compared to band length, the propagation criterion becomes

$$\frac{\tau_\infty - \tau_r}{\tau_p - \tau_r} = \sqrt{\frac{4}{\pi(1-\nu)} \frac{G}{\tau_p - \tau_r} \frac{\bar{\delta}}{\ell}} \tag{41}$$

for the case shown in fig.7, where (eqs.35 and 8) are used.

ESTIMATE OF SIZE OF END REGION

The J-integral has led to calculations of the drving force and propagation criterion. It is however not possible to obtain further information such as the size of the end region at failure without fairly elaborate calculations. This is due in part to the nonlinear τ,δ relation which must be imposed as a boundary condition. We will therefore estimate the size of the end region approximately by assuming a distribution of τ with distance from the tip of the shear band, a distribution which contains the end region length ω as a parameter, and calculating from elasticity theory the implied τ,δ curve. If the curve is of a reasonable shape the size ω may then be determined as that which gives the proper value of $\int(\tau - \tau_r)\, d\delta$. Fortunately, an assumed linear variation of stress within the end zone, fig.8a, leads to a reasonable curve for our purposes. We assume that the stress intensity factor induced by the applied loadings would be K if the residual stress τ_r alone acted along the band. The restraint of the band surfaces by stresses in excess of τ_r has the

effect of inducing a K factor of opposite sign, and the end zone extent ω is to be chosen so that there is no net stress singularity at the tip. The magnitude of the stress intensity factor induced by the excess shear stresses is (Rice 1968b)

$$\sqrt{\frac{2}{\pi}} \int_0^{\omega} \frac{\tau(R) - \tau_r}{\sqrt{R}} \, dR$$

where we use the formula for the effect of surface loadings on a semi-infinite shear band or crack. Writing

$$\tau - \tau_r = (\tau_p - \tau_r)(1 - \frac{R}{\omega}) \qquad (0 < R < \omega) \tag{42}$$

from the assumed linear variation, we therefore wish to choose ω so that

$$K = \sqrt{\frac{2}{\pi}} (\tau_p - \tau_r) \int_0^{\omega} \frac{(1 - R/\omega)}{\sqrt{R}} \, dR = \frac{4}{3} (\tau_p - \tau_r) \sqrt{\frac{2\omega}{\pi}} \tag{43}$$

Hence

$$\omega = \frac{9\pi}{32} \left(\frac{K}{\tau_p - \tau_r}\right)^2 \tag{44}$$

But we already know from eq.(40) how K^2 must relate to the area under whatever τ, δ curve is implied. Thus the estimated size of the end zone is

$$\omega = \frac{9\pi}{16\,(1-\nu)} \frac{G}{\tau_p - \tau_r} \bar{\delta} \tag{45}$$

Further, by standard calculations of crack elasticity under the loading depicted in fig.8a, one finds that the slip displacements implied by eq.(42) for $\tau - \tau_r$

$$\delta = \frac{9}{8} \bar{\delta} \left\{ (1 + R/\omega) \sqrt{R/\omega} - \frac{1}{2} (1 - R/\omega)^2 \log \frac{1 + \sqrt{R/\omega}}{1 - \sqrt{R/\omega}} \right\} \tag{46}$$

This may be plotted as a function of $(\tau - \tau_r)/(\tau_p - \tau_r)$ after elimination of R/ω with eq.(42). The resulting τ, δ curve is plotted in fig.8b. It has the expected form, and is not inconsistent with the model.

If we accept this estimate and put $\nu = 1/4$ and $G/\tau_p = 50$ as earlier, then

$$\omega \approx 125 \frac{\tau_p}{\tau_p - \tau_r} \bar{\delta} \tag{47}$$

If the stress ratio is taken as 2, then the estimated size of the end region ranges from 0.5 m to 2.5 m as $\bar{\delta}$ ranges from 2 to 10 mm. Hence it seems quite conceivable that the assumption of a small end zone in comparison to pertinent geometric dimensions may frequently be valid in natural soil failures, although the condition seems almost impossible to attain in laboratory experiments. Again we see the implication of a size effect in soil mass failures, for the size of the end region ω at failure is set by the material parameter $\bar{\delta}$ more or less independently of the actual size of the mass.

TIME EFFECTS

In the preceding analyses we have imagined that a shear band of a certain length already exists, and have determined a propagation criterion which tells us how large the applied loads have to be if a shear band of that length is to propagate. In the problems which typically arise in soil mechanics, such as slope analysis, the external loads are gravitational and remain more or less constant, though sometimes geometry changes occur, as when the toe of a slope is cut or progressively eroded. We can conjecture that what happens is that a shear band is initiated at a stress concentration, grows slowly until it reaches a critical length, and then propagates rapidly. Our analysis has not explained how the band can grow slowly, or what time effects control how fast this happens.

The simplest and most obvious time effects in soil mechanics are those controlled by the diffusion of pore water within the soil, which allows change of water content and effective stress. In this paper we have examined the consequences of a relationship between τ and δ on a shear band, and of simple stress-strain relations for the soil outside the band, such as the equation relating $\bar{\sigma}_{11}$ to $\bar{\varepsilon}_{11}$ in the slope analysis case. What these relationships are must naturally depend on the drainage conditions, and on whether or not there is time for water content and pore pressure changes to diffuse. Three cases can be distinguished; the model we have studied can be applied to all of them, but the relations between τ and δ and between $\bar{\sigma}_{11}$ and $\bar{\varepsilon}_{11}$ will be different. The discussion is restricted to the case of a shear band parallel to a uniform slope (Figure 5); closely similar conclusions apply to other cases.

Consider as the first case that the shear band advances rapidly in comparison to the time scales for diffusion. The soil deformation is "undrained". Under such conditions the shearing of heavily overconsolidated clay creates negative pressure or suction in the pore fluid. This increases the effective compressive stress transmitted to the soil particles, and would have the effect of increasing the resistance against sliding at the tip of a shear band. In addition, the soil in the overhanging layer above the band responds as a stiffer material than it would be under drained conditions. Hence, from the point of view of the propagation criterion

$$\frac{h}{2E'}\left[(\tau_g - \tau_r)\,\ell/h + p^o\right]^2 = \int(\tau - \tau_r)\,d\delta\ , \qquad (32)\text{ bis}$$

the shorter the time available for diffusion, the more the resistance term on the right is increased and the more the driving force term is decreased.

However, there would seem to be two time scales for diffusion, and as a second case we consider that the speed of propagation is still too rapid to allow drained behaviour of the overhanging layer, but is nevertheless

sufficiently slow that negligible excess suctions are generated in the heavily sheared material near the tip of the band. In this case E' is the same as for the first case, but now the resistance term has settled to a fixed lower value appropriate to drained behaviour on the small size scale of the band thickness. Indeed, it is possible to make estimates of the elevation of the τ, δ curve induced by a given speed of advance, and hence to determine from (32) a propagation speed governed by pore water diffusion. We intend to describe this in a subsequent paper.

In the third case the deformation is wholly drained, and the relation between $\bar{\sigma}_{11}$ and $\bar{\varepsilon}_{11}$ is that for a drained material, so that E' is reduced below its value for the first and second cases, while the value of $\int(\tau - \tau_r)\, d\delta$ is the same as in the second case. This corresponds to much slower propagation: pore-pressure changes in clay diffuse so slowly over distances of the order of several meters (a typical depth to a shear band) that they may require time of the order of 10^2 years.

We may consider this time scale for bulk drainage in relation to the propagation speed by noting that the stress $\bar{\sigma}_{11}$ in the layer changes from $-p^o$ to $(\tau_g - \tau_r)\, \ell/h$ over a distance of the order of the end zone size ω. Hence the time scale over which the material responds is ω divided by the speed of advance, and material properties such as E', which appear in the propagation criterion should be chosen as appropriate to this time scale. This becomes clear when we note that for inelastic behaviour it is the integral of $\sigma_{ij}\, \partial\varepsilon_{ij}/\partial x_1$ over the deforming region which enters the J integral as a difference in W.

As well as time effects associated with water diffusion, there can be expected to be viscoelastic deformations, especially creep, which occur even if there are no pore pressure changes (Bjerrum, 1967b; Bishop, 1968b). In this event the material properties are again to be chosen as those

appropriate to the time scale ω (propagation speed) so that, for example, E' can be approximately regarded as the viscoelastic modulus at this deformation time. The modulus reduces with increasing time, ultimately to zero for a Maxwell model, and this means that the effective driving force term will be increased. It is possible that creep-like effects could affect the resistance term as well by progressive degradation of the strength at a given δ. There is also the possibility of aging effects, such as the weathering break-down of soil bonds (Bjerrum, 1967a), which may be considered to increase the energy made available by release of lateral pressure.

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Appendix: Energy Rate Interpretation

This section follows the lines of Rice's (1968b) proof that J is the energy release rate for crack extension in elastic bodies, an interpretation which relates closely to Eshelby's (1956) earlier use of the integral in the computation of energetic 'forces' on point or line defects in solids.

We assume for the present that the material outside the shear band is elastic. Let ℓ be the shear band length in fig.2, as measured from the origin of the x_1 axis. Define G' as the energy surplus made available per unit of quasi-static band advance, this being defined as the excess of the work of external forces over the energy stored in deformation and the dissipation against the residual part of the shear strength in the band. Hence if point P lies outside the end region,

$$G' = \int_{\Gamma} T_i \frac{du_i}{d\ell} ds + \int_{A} f_i \frac{du_i}{d\ell} dA - \frac{d}{d\ell} \int_{A} W dA - \int_{P}^{T} \tau_r \frac{d\delta}{d\ell} dx_1 \qquad \text{(A-1)}$$

Here A is the area enclosed by Γ. The choice of P and Γ is arbitrary, for the virtual work theorem assures cancellation of the contributions from the annular region between any two Γ choices both lying outside the end zone. By the same theorem it is obvious that for a quasi-static advance of the band

$$G' = \int_{P}^{T} (\tau - \tau_r) \frac{d\delta}{d\ell} dx_1 , \qquad \text{(A-2)}$$

the latter being the dissipation against shear resistance in excess of the residual strength.

Now any field variable $f = f(x_1, x_2, \ell)$ can equally well be written in the form $f(x_1', x_2, \ell)$, with $x_1' = x_1 - \ell$, as seen by a moving observer. Hence

$$\frac{df}{d\ell} = - \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial \ell} \qquad \text{(A-3)}$$

the notation $\partial/\partial \ell$ being the derivative computed by the moving observer. In this notation

$$\frac{d}{d\ell}\int_A W dA = -\int_\Gamma W dx_2 + \int_A \frac{\partial W}{\partial \ell} dA$$

$$\int_A f_i \frac{du_i}{d\ell} dA = -\int_\Gamma f_i u_i \, dx_2 + \int_A f_i \frac{\partial u_i}{\partial \ell} dA$$

$$\int_\Gamma T_i \frac{du_i}{d\ell} ds = -\int_\Gamma T_i \frac{\partial u_i}{\partial x_1} ds + \int_\Gamma T_i \frac{\partial u_i}{\partial \ell} ds \tag{A-4}$$

$$\int_P^T \tau_r \frac{d\delta}{d\ell} dx_1 = \tau_r \delta_P + \int_P^T \tau_r \frac{\partial \delta}{\partial \ell} dx_1$$

where it is assumed that the material properties and body forces are independent of x_1. Thus upon recalling (A-1)

$$G' = J_P - \tau_r \, \delta_P + \int_\Gamma T_i \frac{\partial u_i}{\partial \ell} ds + \int_A f_i \frac{\partial u_i}{\partial \ell} dA - \int_A \frac{\partial W}{\partial \ell} dA - \int_P^T \tau_r \frac{\partial \delta}{\partial \ell} dx_1 \tag{A-5}$$

All of the terms containing $\partial/\partial \ell$ vanish when the deformations as viewed by the moving observer are fixed and hence $G' = J_P - \tau_r \, \delta_P$ in that case.

The same result is also true in the model appropriate to a small end zone, for which it is assumed that $\tau = \tau_r$ all along the shear band, with a singularity resulting in the elastic field at its tip. In that case the virtual work theorem (which could not be applied to the virtual displacement $du_i/d\ell$, because the integral of $dW/d\ell$ is then divergent) requires that all the terms involving $\partial/\partial \ell$ sum to zero, so that again $G' = J_P - \tau_r \, \delta_P$.

More generally, with a finite end zone, the $\partial/\partial \ell$ terms would sum to zero if τ rather than τ_r appeared in the last. Hence

$$G' = J_P - \tau_r \, \delta_P + \int_P^T (\tau - \tau_r) \frac{\partial \delta}{\partial \ell} dx_1 \tag{A-6}$$

On the other hand, the formalism of (A-3) changes (A-2) to

$$G' = \int (\tau - \tau_r)\, d\delta + \int_P^T (\tau - \tau_r) \frac{\partial \delta}{\partial \ell}\, dx_1 \tag{A-7}$$

We thus see from the first of these that $J_P - \tau_r\, \delta_P$ is not <u>always</u> equal to the energy surplus. But the second makes it clear that in the same circumstances the required energy surplus is not just simply $\int (\tau - \tau_r)\, d\delta$. The identical additional term appearing in each equation means that

$$J_P - \tau_r\, \delta_P = \int (\tau - \tau_r)\, d\delta \tag{A-8}$$

always, regardless of the validity or not of the energy interpretations for the separate terms. Indeed, this result was derived in the text independently of these interpretations.

When the material outside the shear band is <u>inelastic</u>, the rate of change of strain energy in (A-1) may be replaced with

$$\int_A \sigma_{ij} \frac{d\varepsilon_{ij}}{d\ell}\, dA$$

which is the net rate of energy storage and/or dissipation by deformation, and the same interpretation of G' as an energy surplus remains. Then the first member of (A-1) becomes

$$\int_A \sigma_{ij} \frac{d\varepsilon_{ij}}{d\ell}\, dA = - \int_\Gamma W\, dx_2 + \int_A \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial \ell}\, dA \tag{A-9}$$

provided that W is defined by integrating $\partial W/\partial x_1 = \sigma_{ij}\, \partial \varepsilon_{ij}/\partial x_1$. This leads directly to (A-5) with W as here defined in the formula for J, and with $\partial W/\partial \ell$ replaced by $\sigma_{ij}\, \partial \varepsilon_{ij}/\partial \ell$. Hence, whenever the end-region deformations appear unchanged to the moving observer, $J_P - \tau_r\, \delta_P$ is the energy surplus for dissipation against strengths in excess of the residual value even if the material behaviour is inelastic outside the band. It is difficult to pursue this interpretation for the model in which $\tau = \tau_r$ all along the band

and a singularity appears in the continuum field at the tip. Then the necessary term $\int_A \sigma_{ij}(d\epsilon_{ij}/d\ell)\, dA$ for an inelastic material, in contrast to $(d/d\ell)\int_A W dA$ for an elastic material, involves an integral which may be at least formally divergent at the shear band tip. An analogous difficulty in interpretation would arise for an inelastic tensile crack model which included no cohesive zone at the tip.

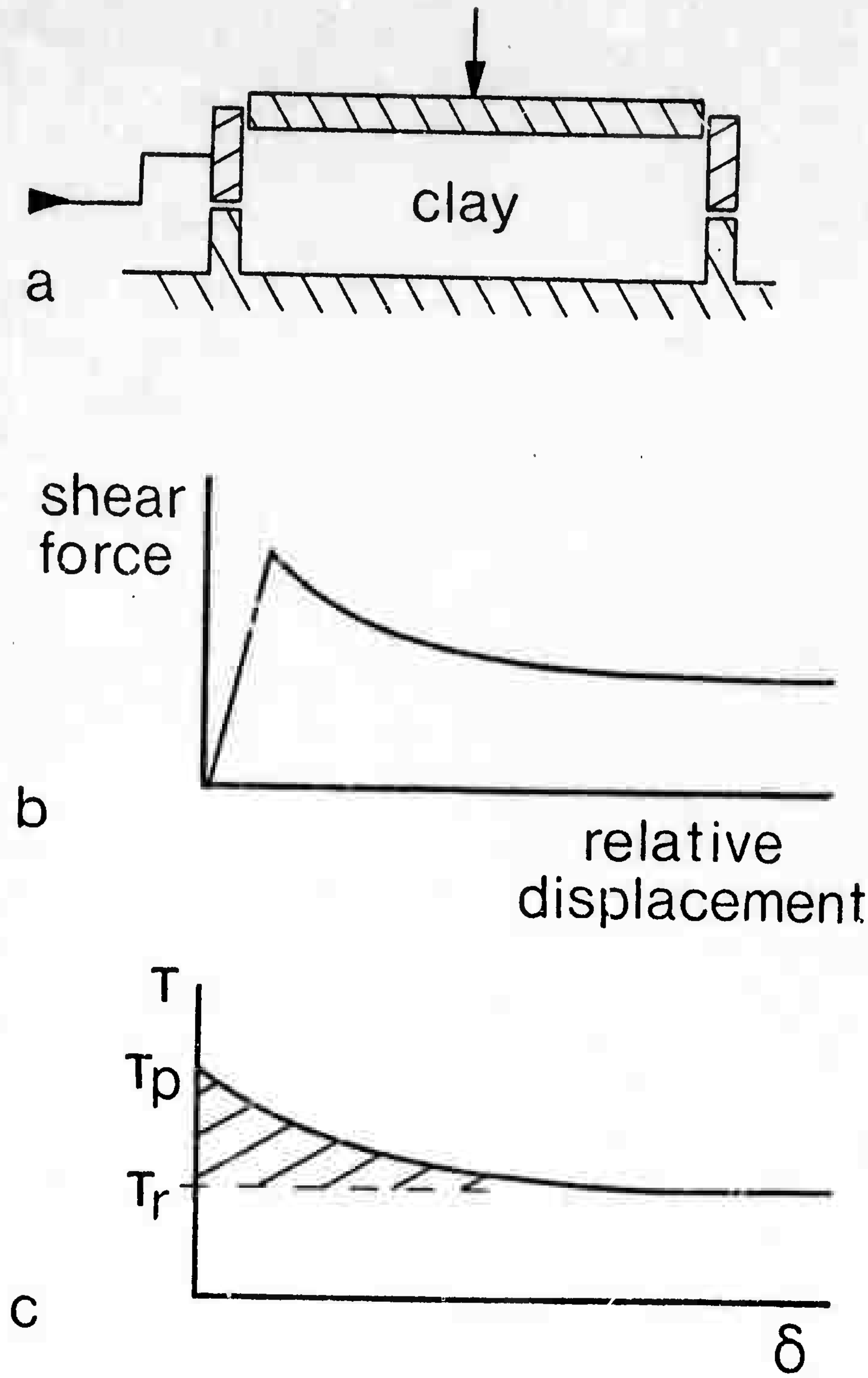


FIGURE 1

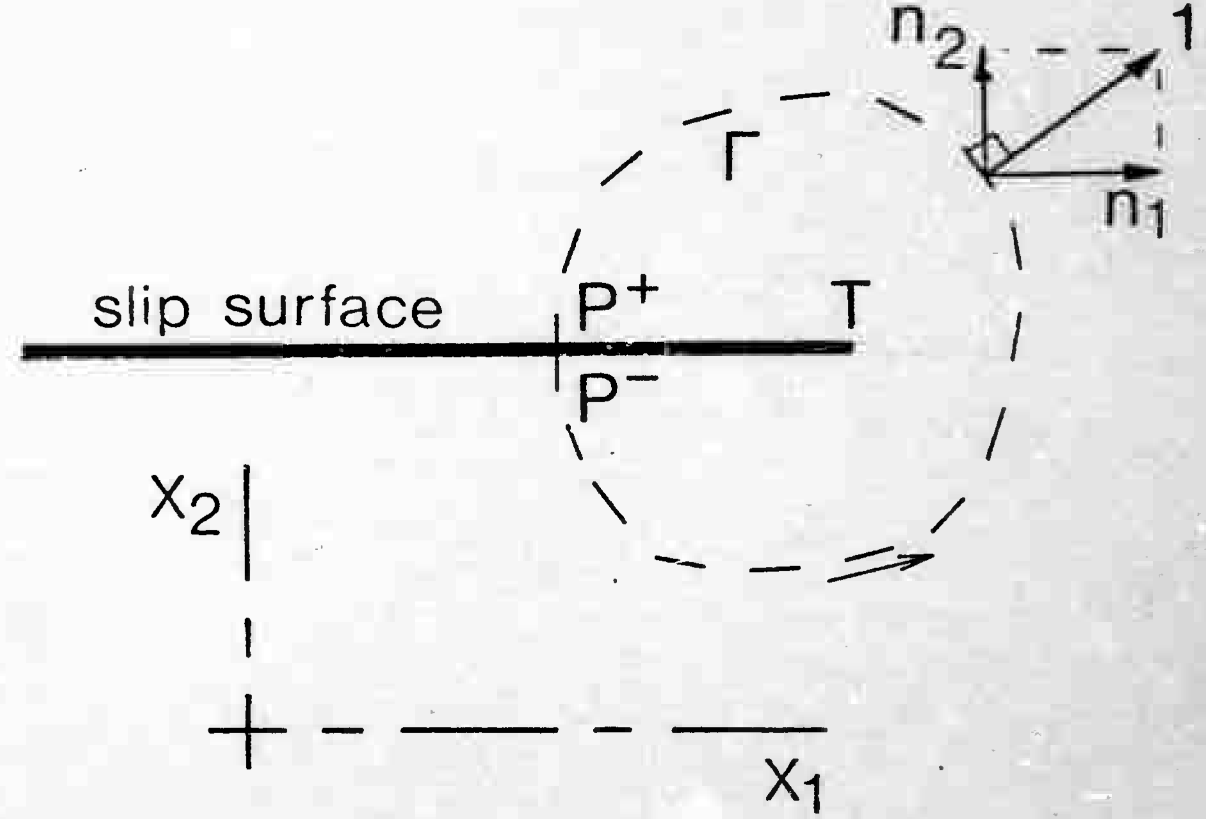


FIGURE 2

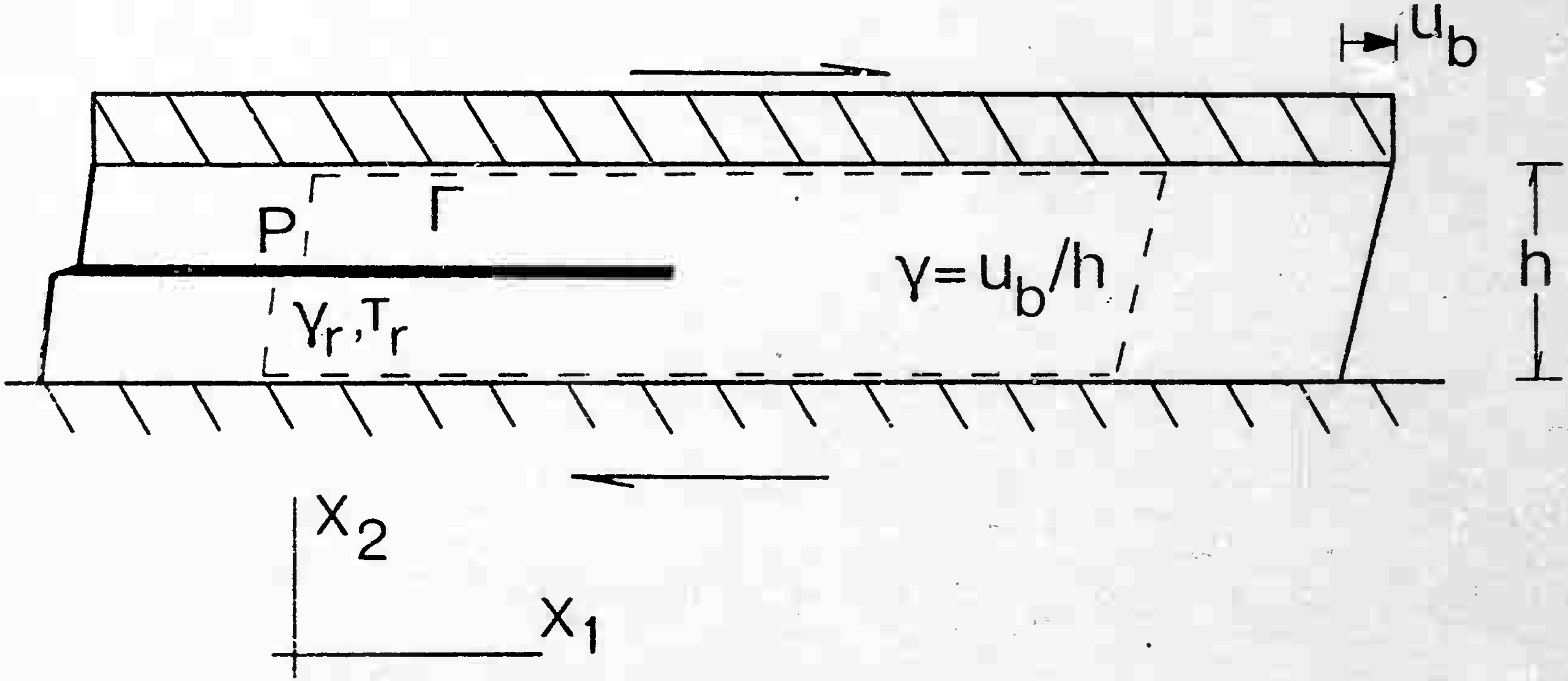


FIGURE 3

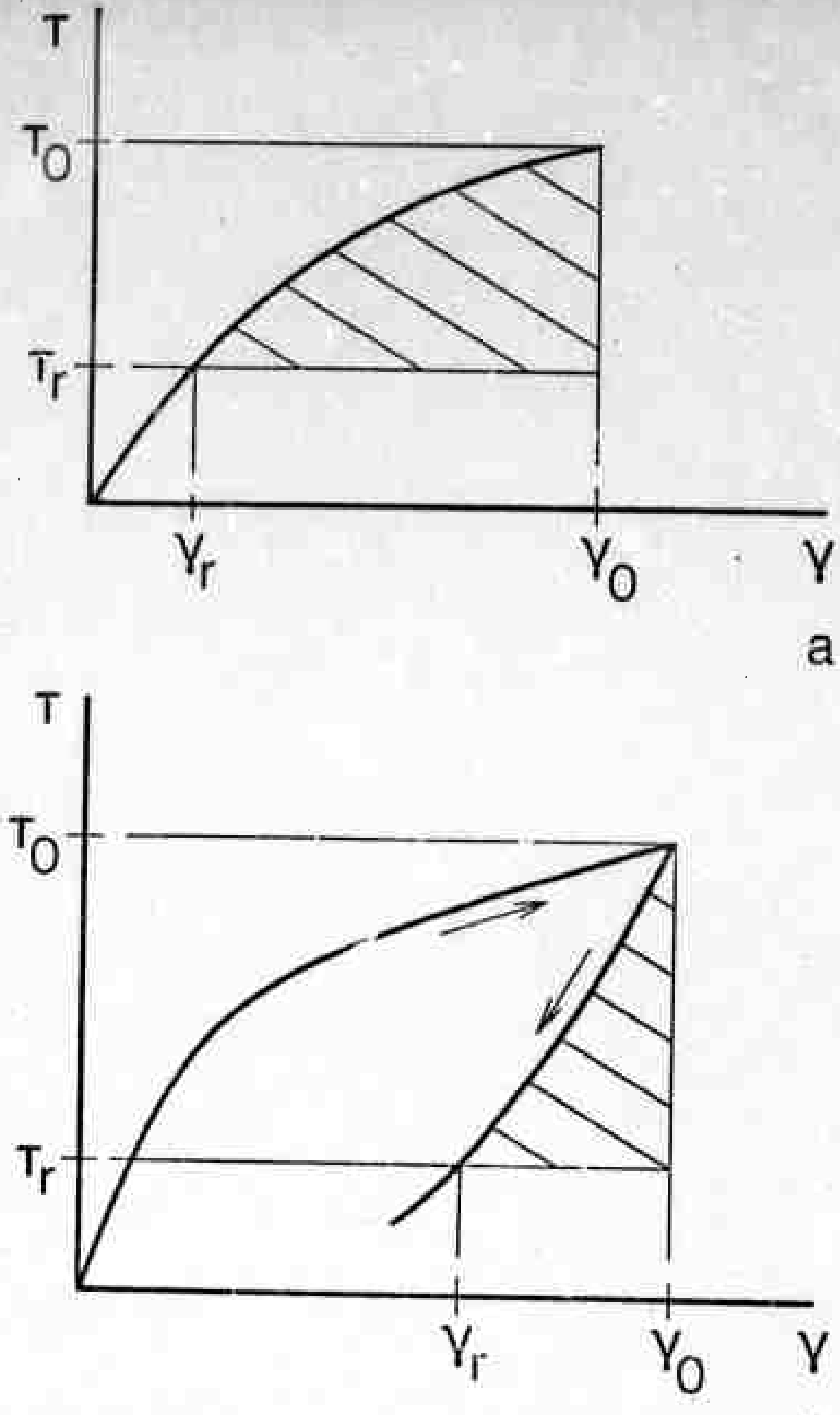
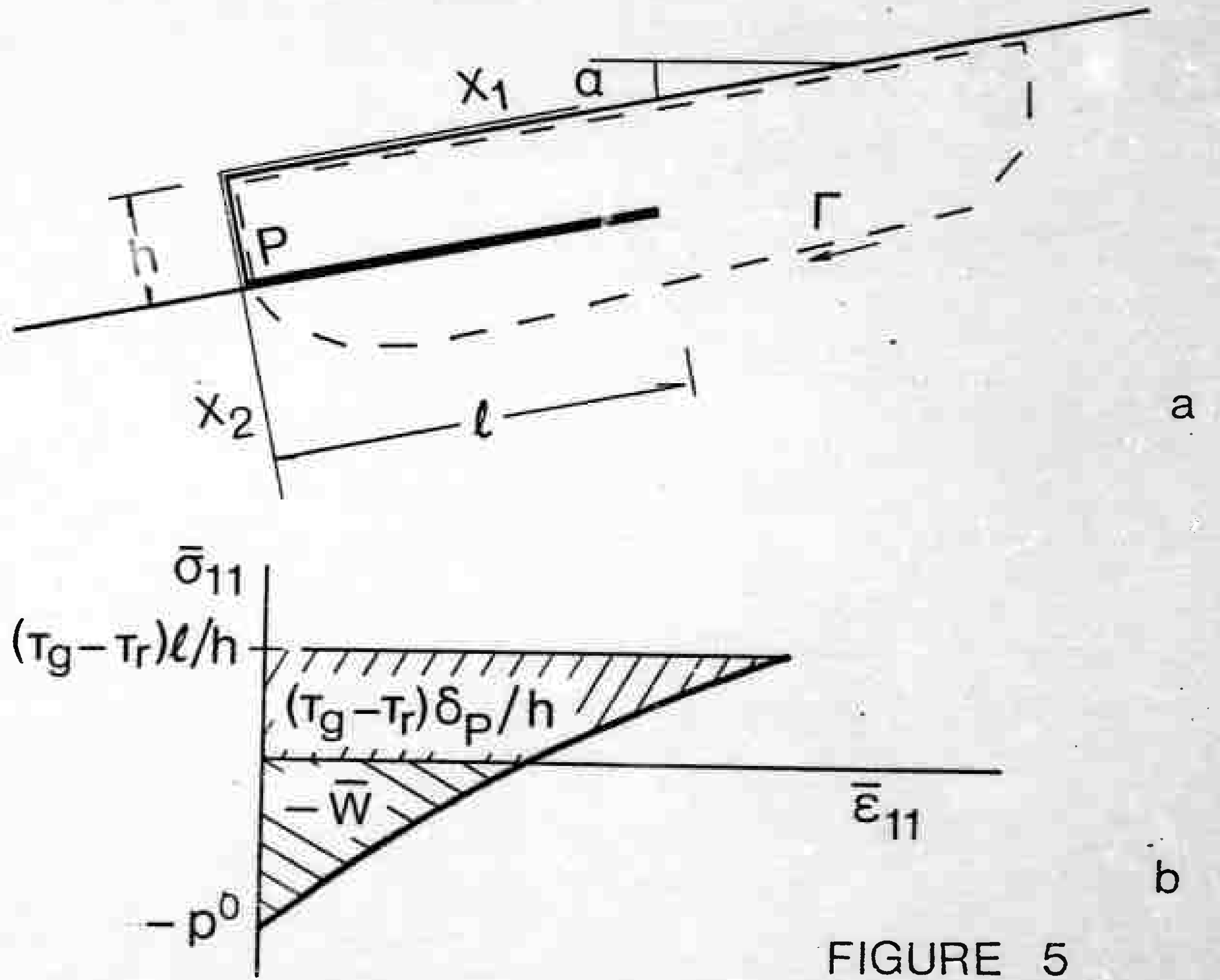


FIGURE 4



X_1
α
h
P
Γ
X_2
ℓ
a
$\bar{\sigma}_{11}$
$(\tau_g - \tau_r)\ell/h$
$(\tau_g - \tau_r)\delta_P/h$
$-\bar{W}$
$\bar{\varepsilon}_{11}$
$-p^0$
b

FIGURE 5

A-9

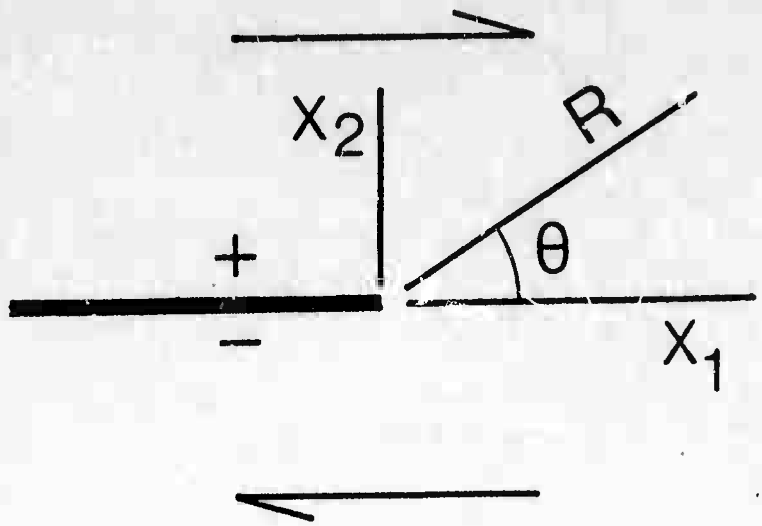


FIGURE 6

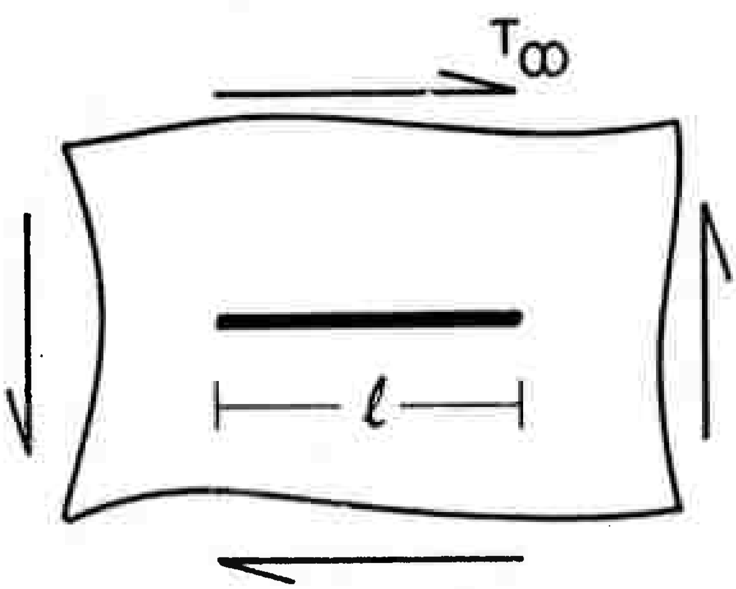


FIGURE 7

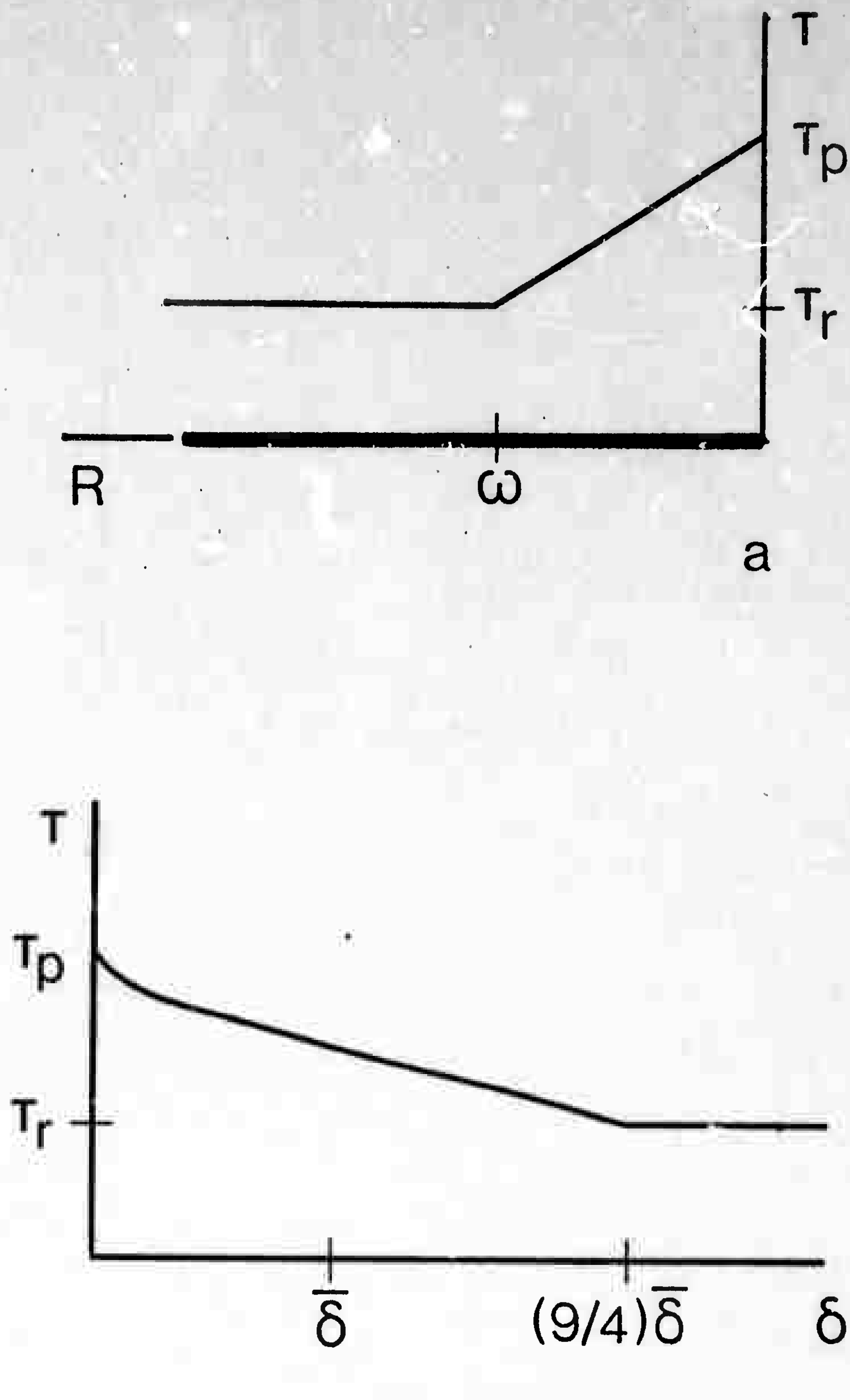


FIGURE 8